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Stability Derivatives for a Slender Missile with Application to a Wing-Body-Vertical-Tail Configuration

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ABSTRACT

Theoretical stability derivatives are determined for a slender missile through use of the concept of inertia coefficients of its cross section. The inertia coefficients are determined by the velocity potentials for two-dimensional motion of the cross section in two mutually perpendicular directions and for rotation about an axis perpendicular to the cross section. The inertia coefficients are derived for the cross section of a configuration consisting of a slender body of revolution with low aspect ratio wings and unequal vertical tails.

INTRODUCTION

Munk,[1] Jones,[2] and Spreiter[3] have shown how to estimate the aerodynamic forces on slender fuselages, wings, and wing-fuselage combinations by considering the flow in planes perpendicular to the wing or fuselage axis to be uninfluenced by the change in the component of fluid velocity parallel to this axis. Munk and Jones made use of the concept of the apparent additional mass of the cross section in a two-dimensional flow to find the lift and moment due to angle of attack. This concept is enlarged upon here to include all six of the inertia coefficients of the cross section, utilizing them to find all the stability derivatives (except the axial force derivatives) of the three-dimensional configuration.

Consideration is limited here to configurations where the aft end of the fuselage and all wing trailing edges lie in a plane perpendicular to the long axis of the configuration. Extension to cases where trailing edges occur ahead of this base plane can be made by considering the effect of the vorticity shed from these edges as shown by Jones.

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INERTIA COEFFICIENTS OF THE CROSS SECTION

Consider a slender body† moving through an infinite fluid medium. By "slender" we shall mean that the maximum dimension of any cross section of the body is small compared to the length of the body and that the dimensions of the cross section change slowly with distance along the length of the body. Let x,y,z be axes fixed relative to the body with origin at the center of gravity of the body and with the x-axis in the direction of the length of the body (see Fig. 1). We consider only those motions of the body where the instantaneous velocity vector of any point in the body makes a small angle with the body x-axis. Let ξ,η,ζ be axes such that

† We shall use the term "body," henceforth, to mean a general aerodynamic shape that includes wings, fuselages, wing-fuselage combinations, etc.

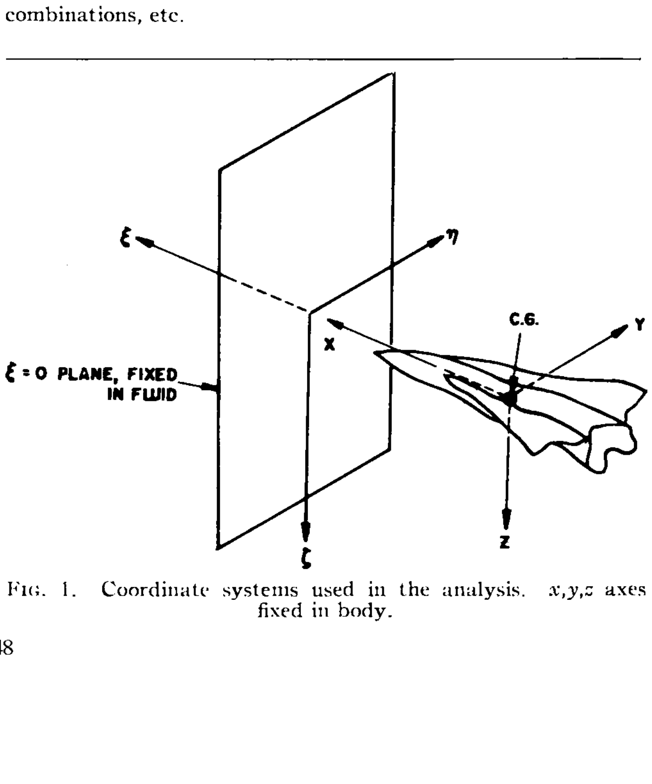

FIG. 1. Coordinate systems used in the analysis. x,y,z axes fixed in body.

the $\xi = 0$ plane is a plane at rest with respect to the fluid far away from the body and such that the ξ-axis is parallel to the x-axis at the instant under consideration. Let the η,ζ-axes be the projections of the y,z-axes onto the $\xi = 0$ plane.

We approximate the three-dimensional fluid flow around the body by considering the fluid in the η-ζ plane to be incompressible and the fluid motion to be two-dimensional—i.e., uninfluenced by any part of the body or fluid which does not lie in the η-ζ plane. The motion in the η-ζ plane is thus a nonstationary motion induced by a two-dimensional body (the three-dimensional body's cross section) whose velocities and shape change with time.

The kinetic energy of the fluid in the η-ζ plane (actually a kinetic energy per unit distance normal to the η-ζ plane) is given by[4]

$$T' = \frac{1}{2}\rho S[v, w, pD][A]\begin{bmatrix} v \\ w \\ pD \end{bmatrix} \tag{1}$$

where the symmetric matrix $[A]$ is given by

$$[A] = \begin{bmatrix} A_{11}, & A_{12}, & A_{13} \\ A_{12}, & A_{22}, & A_{23} \\ A_{13}, & A_{23}, & A_{33} \end{bmatrix} = -\frac{1}{S}\begin{bmatrix} \oint \varphi_1 \frac{\partial \varphi_1}{\partial n} ds, & \oint \varphi_1 \frac{\partial \varphi_2}{\partial n} ds, & \frac{1}{D}\oint \varphi_1 \frac{\partial \varphi_3}{\partial n} ds \\ \oint \varphi_1 \frac{\partial \varphi_2}{\partial n} ds, & \oint \varphi_2 \frac{\partial \varphi_2}{\partial n} ds, & \frac{1}{D}\oint \varphi_2 \frac{\partial \varphi_3}{\partial n} ds \\ \frac{1}{D}\oint \varphi_1 \frac{\partial \varphi_3}{\partial n} ds, & \frac{1}{D}\oint \varphi_2 \frac{\partial \varphi_3}{\partial n} ds, & \frac{1}{D_2}\oint \varphi_3 \frac{\partial \varphi_3}{\partial n} ds \end{bmatrix} \tag{2}$$

and

S = a reference area

D = a reference length

ρ = fluid density

φ_1 = potential due to a unit velocity of the cross section in the η direction

φ_2 = potential due to a unit velocity of the cross section in the ζ direction

φ_3 = potential due to a unit angular velocity of the cross section about the ξ-axis

v = velocity of the cross section in the η direction

w = velocity of the cross section in the ζ direction

p = angular velocity of the cross section about the ξ-axis

$\oint (\ )ds$ = the line integral around the boundary of the cross section

The dimensionless quantities A_{ij} we shall call the inertia coefficients of the cross section.

The Forces Acting on the Body Cross Section

The aerodynamic sideforce, downforce, and rolling moment per unit length on the body at any given cross section are given by[4]

$$\begin{bmatrix} Y' \\ Z' \\ l' \end{bmatrix} = -\frac{d}{dt}\begin{bmatrix} \frac{\partial T'}{\partial v} \\ \frac{\partial T'}{\partial w} \\ \frac{\partial T'}{\partial p} \end{bmatrix} + p\begin{bmatrix} \frac{\partial T'}{\partial w} \\ -\frac{\partial T'}{\partial v} \\ 0 \end{bmatrix} + w\begin{bmatrix} 0 \\ 0 \\ \frac{\partial T'}{\partial v} \end{bmatrix} - v\begin{bmatrix} 0 \\ 0 \\ \frac{\partial T'}{\partial w} \end{bmatrix} \tag{3}$$

The yawing moment and pitching moment per unit length are given by

$$\begin{bmatrix} n' \\ -m' \end{bmatrix} = \begin{bmatrix} xY' \\ xZ' \end{bmatrix} \tag{4}$$

Now T' is a function of v, w, p and the six inertia coefficients. Thus T' can change with time in two ways: (1) by a change in the motion of the cross section and (2) by a change in the shape of the cross section. Furthermore, the motion of the cross section can change in two ways: (1) by a change in the linear velocity of the c.g. of the body and (2) by a change in the angular velocity of the body. Thus,

$$\left.\begin{aligned} v &= \beta U + rx \\ w &= \alpha U - qx \end{aligned}\right\} \tag{5}$$

where

α = angle of attack of the body } linear velocities
β = angle of sideslip of the body }

q = pitching angular velocity of the body } angular velocities
r = yawing angular velocity of the body }

U = forward velocity of the body

All five of these quantities are functions of the time alone.

The inertia coefficients are functions of x, since the shape of the cross section changes with x. Therefore,

$$(d/dt) = (\partial/\partial t) - U(\partial/\partial x) \tag{6}$$

Since $[A]$ is a symmetric matrix,

$$\frac{d}{dt}\begin{bmatrix} \dfrac{\partial T'}{\partial v} \\ \dfrac{\partial T'}{\partial w} \\ \dfrac{1}{D}\dfrac{\partial T'}{\partial p} \end{bmatrix} = \rho S\left[\frac{dA}{dt}\right]\begin{bmatrix} v \\ w \\ pD \end{bmatrix} + \rho S[A]\begin{bmatrix} \dfrac{dv}{dt} \\ \dfrac{dw}{dt} \\ D\dfrac{dp}{dt} \end{bmatrix} \tag{7}$$

From Eqs. (5) and (6) we see that

$$\begin{bmatrix} \dfrac{dv}{dt} \\ \dfrac{dw}{dt} \\ D\dfrac{dp}{dt} \end{bmatrix} = \begin{bmatrix} \dot\beta U + \beta\dot U + \dot r x - Ur \\ \dot\alpha U + \alpha\dot U - \dot q x + Uq \\ D\dot p \end{bmatrix} \tag{8}$$

Substituting Eqs. (5), (6), (7), and (8) into Eq. (3), we have, finally,

$$\frac{D}{\rho U^2 S}\begin{bmatrix} Y' \\ Z' \\ \dfrac{l'}{D} \end{bmatrix} = -[A]\begin{bmatrix} \dfrac{\dot\beta D}{U} + \beta\dfrac{\dot U D}{U^2} + \dfrac{\dot r D^2}{U^2}\dfrac{x}{D} \\ \dfrac{\dot\alpha D}{U} + \alpha\dfrac{\dot U D}{U^2} - \dfrac{\dot q D^2}{U^2}\dfrac{x}{D} \\ \dfrac{\dot p D^2}{U^2} \end{bmatrix} + \frac{\partial}{\partial\left(\dfrac{x}{D}\right)}\left\{[A]\begin{bmatrix} \beta + \dfrac{rD}{U}\dfrac{x}{D} \\ \alpha - \dfrac{qD}{U}\dfrac{x}{D} \\ \dfrac{pD}{U} \end{bmatrix}\right\} - \left\{\left(\beta + \frac{rD}{U}\frac{x}{D}\right)[E] + \left(\alpha - \frac{qD}{U}\frac{x}{D}\right)[F] + \frac{pD}{U}[G]\right\}\begin{bmatrix} \beta + \dfrac{rD}{U}\dfrac{x}{D} \\ \alpha - \dfrac{qD}{U}\dfrac{x}{D} \\ \dfrac{pD}{U} \end{bmatrix} \tag{9}$$

$$\left.\begin{aligned} [E] &= \begin{bmatrix} 0, & 0, & 0 \\ 0, & 0, & 0 \\ A_{12}, & A_{22}, & A_{23} \end{bmatrix} \\ [F] &= -\begin{bmatrix} 0, & 0, & 0 \\ 0, & 0, & 0 \\ A_{11}, & A_{12}, & A_{13} \end{bmatrix} \\ [G] &= \begin{bmatrix} -A_{12}, & -A_{22}, & -A_{23} \\ A_{11}, & A_{12}, & A_{13} \\ 0, & 0, & 0 \end{bmatrix} \end{aligned}\right\} \tag{10}$$

The Total Forces and Moments on the Body

By integrating Eqs. (9) and (4) with respect to x over the length of the body, the sideforce Y, the downforce Z, the rolling moment l, the pitching moment m, and the yawing moment n can be found to second order in terms of α, β, p, q, r, $\dot\alpha$, $\dot\beta$, $\dot p$, $\dot q$, $\dot r$, and $\dot U$.

We shall define force and moment coefficients as follows:

$$[C_Y, C_Z, C_l, C_m, C_n] = \frac{D}{(1/2)\rho U^2 S}\int_{-x_b/D}^{x_n/D}\left[Y', Z', \frac{l'}{D}, \frac{m'}{D}, \frac{n'}{D}\right] d\left(\frac{x}{D}\right) \tag{11}$$

where x_n = distance from the c.g. to the nose of the body and x_b = distance from the c.g. to the base plane of the body.

Integrating Eqs. (9) and (4) and using the definitions of Eq. (11), we have, finally,

$$\begin{bmatrix} C_y \\ C_z \\ C_l \end{bmatrix} = -2[\bar{A}]\begin{bmatrix} \beta - \frac{rD}{U}\frac{x_b}{D} \\ \alpha + \frac{qD}{U}\frac{x_b}{D} \\ \frac{pD}{U} \end{bmatrix} - 2\int_{-x_b/D}^{x_n/D} [A]\, d\left(\frac{x}{D}\right)\begin{bmatrix} \frac{\dot{\beta}D}{U} + \beta\frac{\dot{U}D}{U^2} \\ \frac{\dot{\alpha}D}{U} + \alpha\frac{\dot{U}D}{U^2} \\ \frac{\dot{p}D^2}{U^2} \end{bmatrix} - 2\int_{-x_b/D}^{x_n/D} [A]\frac{x}{D}\, d\left(\frac{x}{D}\right)\begin{bmatrix} \frac{\dot{r}D^2}{U^2} \\ -\frac{\dot{q}D^2}{U^2} \\ 0 \end{bmatrix} -$$

$$2\int_{-x_b/D}^{x_n/D}\left\{\left(\beta + \frac{rD}{U}\frac{x}{D}\right)[E] + \left(\alpha - \frac{qD}{U}\frac{x}{D}\right)[F] + \frac{pD}{U}[G]\right\} d\left(\frac{x}{D}\right)\begin{bmatrix} \beta \\ \alpha \\ \frac{pD}{U} \end{bmatrix} - 2\int_{-x_b/D}^{x_n/D}\left\{\left(\beta + \frac{rD}{U}\frac{x}{D}\right)[E] + \right.$$

$$\left.\left(\alpha - \frac{qD}{U}\frac{x}{D}\right)[F] + \frac{pD}{U}[G]\right\}\frac{x}{D}\, d\left(\frac{x}{D}\right)\begin{bmatrix} \frac{rD}{U} \\ -\frac{qD}{U} \\ 0 \end{bmatrix} \qquad (12)$$

$$\begin{bmatrix} C_n \\ -C_m \\ C_0 \end{bmatrix} = 2\frac{x_b}{D}[\bar{A}]\begin{bmatrix} \beta - \frac{rD}{U}\frac{x_b}{D} \\ \alpha + \frac{qD}{U}\frac{x_b}{D} \\ \frac{pD}{U} \end{bmatrix} - 2\int_{-x_b/D}^{x_n/D} [A]\, d\left(\frac{x}{D}\right)\begin{bmatrix} \beta \\ \alpha \\ \frac{pD}{U} \end{bmatrix} -$$

$$2\int_{-x_b/D}^{x_n/D} [A]\left(\frac{x}{D}\right) d\left(\frac{x}{D}\right)\begin{bmatrix} \frac{\dot{\beta}D}{U} + \frac{rD}{U} + \beta\frac{\dot{U}D}{U^2} \\ \frac{\dot{\alpha}D}{U} - \frac{qD}{U} + \alpha\frac{\dot{U}D}{U^2} \\ \frac{\dot{p}D^2}{U^2} \end{bmatrix} - 2\int_{-x_b/D}^{x_n/D} [A]\left(\frac{x}{D}\right)^2 d\left(\frac{x}{D}\right)\begin{bmatrix} \frac{\dot{r}D^2}{U^2} \\ -\frac{\dot{q}D^2}{U^2} \\ 0 \end{bmatrix} -$$

$$2\int_{-x_b/D}^{x_n/D} [G]\left(\frac{x}{D}\right) d\left(\frac{x}{D}\right)\begin{bmatrix} \frac{pD}{U}\beta \\ \frac{pD}{U}\alpha \\ \left(\frac{pD}{U}\right)^2 \end{bmatrix} - 2\int_{-x_b/D}^{x_n/D} [G]\left(\frac{x}{D}\right)^2 d\left(\frac{x}{D}\right)\begin{bmatrix} \frac{pD}{U}\frac{rD}{U} \\ -\frac{pD}{U}\frac{qD}{U} \\ 0 \end{bmatrix} \qquad (13)$$

where $(^-)$ indicates the value in the base plane and the quantity C_0 has no physical significance.
Example: Consider the flat delta wing of span $\bar{b}$ and chord c. Here,

$$[A] = \begin{bmatrix} 0, & 0, & 0 \\ 0, & \frac{\pi}{4}\frac{b^2}{S}, & 0 \\ 0, & 0, & \frac{\pi}{64}\frac{b^4}{SD^2} \end{bmatrix} \quad \text{where} \quad \begin{cases} b = \bar{b}[(x_n - x)/c] \\ c = x_n + x_b \end{cases} \qquad (14)$$

For simplicity we let $S = (\pi/4)\bar{b}^2$ and $D = \bar{b}$; then,

$$[A] = \begin{bmatrix} 0, & 0, & 0 \\ 0, & \left(\frac{b}{\bar{b}}\right)^2, & 0 \\ 0, & 0, & \left(\frac{b}{2\bar{b}}\right)^4 \end{bmatrix}, \quad [E] = \begin{bmatrix} 0, & 0, & 0 \\ 0, & 0, & 0 \\ 0, & \left(\frac{b}{\bar{b}}\right)^2, & 0 \end{bmatrix} \quad [G] = \begin{bmatrix} 0, & -\left(\frac{b}{\bar{b}}\right)^2, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & 0 \end{bmatrix} \tag{15}$$

and $[F]$ vanishes.

Carrying out the integrations indicated, we find, from Eqs. (12) and (13),

$$\left.\begin{aligned}
C_y &= \frac{2}{3}\frac{c}{D}\frac{pD}{U}\alpha - K_1\frac{pD}{U}\frac{qD}{U} \\
C_z &= -2\alpha - 2\frac{x_b}{D}\frac{qD}{U} - \frac{2}{3}\frac{c}{D}\frac{\dot{\alpha}D}{U} - \frac{2}{3}\frac{c}{D}\alpha\frac{\dot{U}D}{U^2} + K_1\frac{\dot{q}D^2}{U^2} \\
C_l &= -\frac{1}{8}\frac{pD}{U} - \frac{1}{40}\frac{c}{D}\frac{\dot{p}D^2}{U^2} - \frac{2}{3}\frac{c}{D}\alpha\beta - K_1\left(\alpha\frac{rD}{U} - \beta\frac{qD}{U}\right) - K_2\frac{qD}{U}\frac{rD}{U} \\
C_m &= -2\left(\frac{x_b}{D} - \frac{1}{3}\frac{c}{D}\right)\alpha - 2\left(\frac{x_b}{D}\right)^2\frac{qD}{U} - K_1\left(\frac{qD}{U} - \frac{\dot{\alpha}D}{U}\right) + K_1\alpha\frac{\dot{U}D}{U^2} - K_2\frac{\dot{q}D^2}{U^2} \\
C_n &= K_1(pD/U)\alpha - K_2(pD/U)(qD/U)
\end{aligned}\right\} \tag{16}$$

where

$$\begin{cases} K_1 = \frac{1}{6}\left(\frac{c}{D}\right)^2 - \frac{2}{3}\frac{c}{D}\frac{x_b}{D} \\ K_2 = \frac{1}{15}\left(\frac{c}{D}\right)^3 - \frac{1}{3}\left(\frac{c}{D}\right)^2\frac{x_b}{D} + \frac{2}{3}\frac{c}{D}\left(\frac{x_b}{D}\right)^2 \end{cases}$$

These expressions contain all the terms of reference 5 and a few more. It is interesting to note that the sideforce and yawing moment terms that arise due to leading-edge suction are taken care of very simply in this method of analysis.

The Stability Derivatives

Let us denote derivatives in the following manner:

$$[(\ )_\alpha, (\ )_\beta, (\ )_p, (\ )_q, (\ )_r, (\ )_{\dot{\alpha}}, (\ )_{\dot{\beta}}, (\ )_{\dot{p}}, (\ )_{\dot{q}}, (\ )_{\dot{r}}, (\ )_{\dot{U}}] =$$
$$\left[\frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\beta}, \frac{\partial}{\partial\left(\frac{pD}{U}\right)}, \frac{\partial}{\partial\left(\frac{qD}{U}\right)}, \frac{\partial}{\partial\left(\frac{rD}{U}\right)}, \frac{\partial}{\partial\left(\frac{\dot{\alpha}D}{U}\right)}, \frac{\partial}{\partial\left(\frac{\dot{\beta}D}{U}\right)}, \frac{\partial}{\partial\left(\frac{\dot{p}D^2}{U^2}\right)}, \frac{\partial}{\partial\left(\frac{\dot{q}D^2}{U^2}\right)}, \frac{\partial}{\partial\left(\frac{\dot{r}D^2}{U^2}\right)}, \frac{\partial}{\partial\left(\frac{\dot{U}D}{U^2}\right)}\right] \tag{17}$$

All of the derivatives are evaluated at $\alpha = \alpha$, $\beta = p = q = r = \dot{p} = \dot{q} = \dot{r} = \dot{U} = \dot{\alpha} = \dot{\beta} = 0$ (although some other equilibrium condition could, of course, have been specified).

From Eqs. (12), (13), and (17) we see that we can obtain 55 stability derivatives. They are given as follows:

$$\begin{bmatrix} C_{Y_\beta}, & C_{Y_\alpha}, & C_{Y_p} \\ C_{Z_\beta}, & C_{Z_\alpha}, & C_{Z_p} \\ C_{l_\beta}, & C_{l_\alpha}, & C_{l_p} \end{bmatrix} = -2\begin{bmatrix} \bar{A}_{11}, & \bar{A}_{12}, & \bar{A}_{13} \\ \bar{A}_{12}, & \bar{A}_{22}, & \bar{A}_{23} \\ \bar{A}_{13}, & \bar{A}_{23}, & \bar{A}_{33} \end{bmatrix} + 2\alpha\int_{-x_b/D}^{x_n/D}\begin{bmatrix} 0, & 0, & A_{22} \\ 0, & 0, & -A_{12} \\ A_{11}-A_{22}, & 2A_{12}, & A_{13} \end{bmatrix} d\left(\frac{x}{D}\right) \tag{18}$$

$$\begin{bmatrix} -C_{Y_r}, & C_{Y_q} \\ -C_{Z_r}, & C_{Z_q} \\ -C_{l_r}, & C_{l_q} \end{bmatrix} = -2\frac{x_b}{D}\begin{bmatrix} \bar{A}_{11}, & \bar{A}_{12} \\ \bar{A}_{12}, & \bar{A}_{22} \\ \bar{A}_{13}, & \bar{A}_{23} \end{bmatrix} - 2\alpha\int_{-x_b/D}^{x_n/D}\begin{bmatrix} 0, & 0 \\ 0, & 0 \\ A_{11}-A_{22}, & 2A_{12} \end{bmatrix}\frac{x}{D}\, d\left(\frac{x}{D}\right) \tag{19}$$

$$\begin{bmatrix} C_{n_\beta}, & C_{n_\alpha}, & C_{n_p} \\ -C_{m_\beta}, & -C_{m_\alpha}, & -C_{m_p} \end{bmatrix} = 2\frac{x_b}{D}\begin{bmatrix} \bar{A}_{11}, & \bar{A}_{12}, & \bar{A}_{13} \\ \bar{A}_{12}, & \bar{A}_{22}, & \bar{A}_{23} \end{bmatrix} - 2\int_{-x_b/D}^{x_n/D}\begin{bmatrix} A_{11}, & A_{12}, & A_{13} \\ A_{12}, & A_{22}, & A_{23} \end{bmatrix} d\left(\frac{x}{D}\right) - 2\alpha\int_{-x_b/D}^{x_n/D}\begin{bmatrix} 0, & 0, & -A_{22} \\ 0, & 0, & A_{12} \end{bmatrix}\frac{x}{D}\, d\left(\frac{x}{D}\right) \tag{20}$$

$$\begin{bmatrix} -C_{n_r}, & C_{n_q} \\ C_{m_r}, & -C_{m_q} \end{bmatrix} = 2\left(\frac{x_b}{D}\right)^2 \begin{bmatrix} \bar{A}_{11}, & \bar{A}_{12} \\ \bar{A}_{12}, & \bar{A}_{22} \end{bmatrix} + 2\int_{-x_b/D}^{x_n/D} \begin{bmatrix} A_{11}, & A_{12} \\ A_{12}, & A_{22} \end{bmatrix} \frac{x}{D}\, d\left(\frac{x}{D}\right) \tag{21}$$

$$\begin{bmatrix} C_{Y_{\dot\beta}}, & C_{Y_{\dot\alpha}}, & C_{Y_{\dot p}} \\ C_{Z_{\dot\beta}}, & C_{Z_{\dot\alpha}}, & C_{Z_{\dot p}} \\ C_{l_{\dot\beta}}, & C_{l_{\dot\alpha}}, & C_{l_{\dot p}} \end{bmatrix} = -2\int_{-x_b/D}^{x_n/D} \begin{bmatrix} A_{11}, & A_{12}, & A_{13} \\ A_{12}, & A_{22}, & A_{23} \\ A_{13}, & A_{23}, & A_{33} \end{bmatrix} d\left(\frac{x}{D}\right) \tag{22}$$

$$\begin{bmatrix} C_{Y_{\dot r}}, & -C_{Y_{\dot q}} \\ C_{Z_{\dot r}}, & -C_{Z_{\dot q}} \\ C_{l_{\dot r}}, & -C_{l_{\dot q}} \end{bmatrix} = -2\int_{-x_b/D}^{x_n/D} \begin{bmatrix} A_{11}, & A_{12} \\ A_{12}, & A_{22} \\ A_{13}, & A_{23} \end{bmatrix} \frac{x}{D}\, d\left(\frac{x}{D}\right) \tag{23}$$

$$\begin{bmatrix} C_{n_{\dot\beta}}, & C_{n_{\dot\alpha}}, & C_{n_{\dot p}} \\ -C_{m_{\dot\beta}}, & -C_{m_{\dot\alpha}}, & -C_{m_{\dot p}} \end{bmatrix} = -2\int_{-x_b/D}^{x_n/D} \begin{bmatrix} A_{11}, & A_{12}, & A_{13} \\ A_{12}, & A_{22}, & A_{23} \end{bmatrix} \frac{x}{D}\, d\left(\frac{x}{D}\right) \tag{24}$$

$$\begin{bmatrix} C_{n_{\dot r}}, & C_{n_{\dot q}} \\ C_{m_{\dot r}}, & C_{m_{\dot q}} \end{bmatrix} = -2\int_{-x_b/D}^{x_n/D} \begin{bmatrix} A_{11}, & A_{12} \\ A_{12}, & A_{22} \end{bmatrix} \left(\frac{x}{D}\right)^2 d\left(\frac{x}{D}\right) \tag{25}$$

$$\begin{bmatrix} C_{Y_{\dot U}} \\ C_{Z_{\dot U}} \\ C_{l_{\dot U}} \end{bmatrix} = -2\alpha\int_{-x_b/D}^{x_n/D} \begin{bmatrix} A_{12} \\ A_{22} \\ A_{23} \end{bmatrix} d\left(\frac{x}{D}\right) \tag{26}$$

$$\begin{bmatrix} C_{n_{\dot U}} \\ -C_{m_{\dot U}} \end{bmatrix} = -2\alpha\int_{-x_b/D}^{x_n/D} \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} \frac{x}{D}\, d\left(\frac{x}{D}\right) \tag{27}$$

Thus, in general, for a body with a cross section with no symmetries, 14 integrals must be evaluated,

$$\int_{-x_b/D}^{x_n/D} \begin{bmatrix} A_{11}, & A_{12}, & A_{13} \\ 0, & A_{22}, & A_{23} \\ 0, & 0, & A_{33} \end{bmatrix} d\left(\frac{x}{D}\right), \quad \int_{-x_b/D}^{x_n/D} \begin{bmatrix} A_{11}, & A_{12}, & A_{13} \\ 0, & A_{22}, & A_{23} \end{bmatrix} \frac{x}{D}\, d\left(\frac{x}{D}\right), \quad \int_{-x_b/D}^{x_n/D} \begin{bmatrix} A_{11}, & A_{12} \\ 0, & A_{22} \end{bmatrix} \left(\frac{x}{D}\right)^2 d\left(\frac{x}{D}\right)$$

For a cross section with one plane of symmetry (such as almost all present-day airplane and missile configurations), $A_{12} = A_{23} = 0$, and, hence, only nine integrals must be evaluated. For a cross section with two perpendicular planes of symmetry, A_{13} also vanishes and only seven integrals must be evaluated. If the cross section is the same when rotated 90° (such as a cruciform missile configuration), then, also, $A_{11} = A_{22}$ and only four integrals must be evaluated.

It is interesting that the only stability derivatives that require a knowledge of φ_3, the potential due to rolling, are C_{l_p} and $C_{l_{\dot p}}$. Also, because of the symmetry of $[A]$, we find that

$$\begin{aligned} C_{Z_\beta} &= C_{Y_\alpha}, & C_{m_r} &= C_{n_q}, & C_{m_{\dot r}} &= C_{n_{\dot q}}, & C_{Y_{\dot r}} &= C_{n_{\dot\beta}} \\ C_{m_\beta} &= -C_{n_\alpha}, & C_{Z_{\dot\beta}} &= C_{Y_{\dot\alpha}}, & C_{l_{\dot\beta}} &= C_{Y_{\dot p}}, & C_{m_{\dot\alpha}} &= C_{Z_{\dot q}} \\ C_{Z_r} &= -C_{Y_q}, & C_{m_{\dot\beta}} &= -C_{n_{\dot\alpha}} = -C_{Z_{\dot r}} = C_{Y_{\dot q}}; & C_{l_{\dot\alpha}} &= C_{Z_{\dot p}} \end{aligned}$$

and, in addition, when $\alpha = 0$, $C_{l_\beta} = C_{Y_p}$ and $C_{l_\alpha} = C_{Z_p}$.

Application to a Slender Wing-Body-Vertical-Tail Configuration

The method described above is applied here to finding the longitudinal and lateral stability derivatives of a configuration consisting of a slender body of revolution with low aspect ratio wings and unequal low aspect ratio vertical tails (see Fig. 2). The cross section of the configuration is mapped conformally onto a circle in order to determine the velocity potentials for the low-speed two-dimensional motion of the cross section. The inertia coefficients of the cross section are then determined, leading finally to the stability derivatives of the three-dimensional configuration.

Conformal Mapping of the Wing-Body-Vertical-Tail Cross Section onto a Circle†

The cross section of the body is shown in Fig. 3. It consists of a circle with two equal opposing fins and two unequal opposing fins at right angles to the equal fins. The plane of the cross section will be called the w plane. The conformal mapping of the exterior of the finned circle onto a circle is most easily seen in terms of three consecutive simpler mappings:

(1) First the w plane is mapped onto the ξ plane by a Joukowsky transformation that "flattens" the circle,

† The author is indebted to R. H. Edwards for pointing out this transformation to him.

$$\xi = w + (a^2/w)$$

(2) Next the "slit" ξ plane is mapped onto the Z plane by the transformation

$$\xi^2 = Z^2 - c^2$$

(3) Finally, the "flat plate" along the real axis in the Z plane is mapped onto a circle in the ζ plane by a translation and another Joukowsky transformation,

$$Z - \frac{h - f}{2} = \zeta + \frac{(h + f)^2}{16\zeta}$$

Combining these three transformations, the mapping of the finned circle onto a circle is

$$c^2 + \left(w + \frac{a^2}{w}\right)^2 = \left[\frac{h - f}{2} + \zeta + \frac{(h + f)^2}{16\zeta}\right]^2 \quad (28)$$

On the boundary of the contour,

$$\left.\begin{aligned} \zeta &= [(h + f)/4]e^{i\theta} \\ w &= \pm \frac{1}{2}\left[\sqrt{\left(\frac{h - f}{2} + \frac{h + f}{2}\cos\theta\right)^2 - c^2} + \sqrt{\left(\frac{h - f}{2} + \frac{h + f}{2}\cos\theta\right)^2 - c^2 - 4a^2}\right] \end{aligned}\right\} \quad (29)$$

where the plus sign holds for the right half of the w plane and the minus sign holds for the left half of the w plane.

The angles in the ζ plane are given by

$$\left.\begin{aligned} \cos\theta_0 &= 1 - \frac{2(h - \sqrt{c^2 + 4a^2})}{h + f} \\ \cos\theta_1 &= 1 - [2(h - c)/(h + f)] \\ \cos\theta_2 &= 1 - [2f/(h + f)] \\ \cos\theta_3 &= 1 - [2(f - c)/(h + f)] \\ \cos\theta_4 &= 1 - \frac{2(f - \sqrt{c^2 + 4a^2})}{h + f} \end{aligned}\right\} \quad (30)$$

The expressions for c, h, and f in terms of a, s, t_1, and t_2 are

$$\left.\begin{aligned} c &= s[1 - (a^2/s^2)] \\ h &= \sqrt{s^2\left(1 - \frac{a^2}{s^2}\right)^2 + t_1^2\left(1 + \frac{a^2}{t_1^2}\right)^2} \\ f &= \sqrt{s^2\left(1 - \frac{a^2}{s^2}\right)^2 + t_2^2\left(1 + \frac{a^2}{t_2^2}\right)^2} \end{aligned}\right\} \quad (31)$$

It is easily seen that the cross section of the configuration considered can be specialized to twelve particular cases; these are shown in Fig. 4. The mapping can be easily extended to the case of a fuselage with elliptical cross section by first mapping the ellipse onto a circle by a Joukowsky transformation and then proceeding as above.

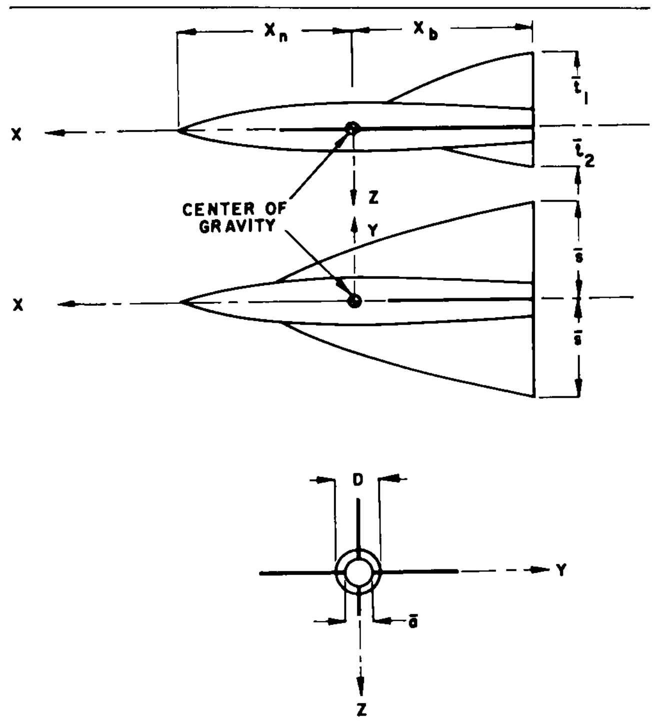

FIG. 2. Slender wing-body-vertical-tail configuration.

THE INERTIA COEFFICIENTS OF THE WING-BODY-VERTICAL-TAIL CROSS SECTION

The expressions for the inertia coefficients of the cross section are given in Eq. (2).

Because of the plane of symmetry of the cross section under consideration here, we see that

$$A_{12} = A_{23} = 0 \quad (32)$$

The vertical fins do not affect the inertia coefficient A_{22}, since they lie along stream lines of the flow. Hence, the case of no vertical fins suffices to find A_{22}. This has already been done by Spreiter.[3] The result is

$$A_{22} = \left(\frac{2S}{D}\right)^2\left(1 - \frac{a^2}{s^2} + \frac{a^4}{s^4}\right) \quad (33)$$

The expression for A_{11} is worked out in Appendix (1). The result is

$$A_{11} = \frac{1}{2}\left(\frac{h + f}{D}\right)^2 [G(\theta_0, \theta_4) + G(\theta_1, \theta_3)] - \left(\frac{2a}{D}\right)^2 \quad (34)$$

where

$$G(\lambda, \mu) = \frac{2}{\pi}\left[1 - \left(\frac{\cos\lambda + \cos\mu}{2}\right)^2\right] \times \left\{K(k)\left[E\left(\frac{\pi - \lambda}{2}, k'\right) + E\left(\frac{\pi - \mu}{2}, k'\right) - \frac{\cos\lambda + \cos\mu}{2\cos\frac{\lambda}{2}\cos\frac{\mu}{2}}\right] - [K(k) - E_0(k)] \times \left[F\left(\frac{\pi - \lambda}{2}, k'\right) + F\left(\frac{\pi - \mu}{2}, k'\right)\right]\right\} \quad (35)$$

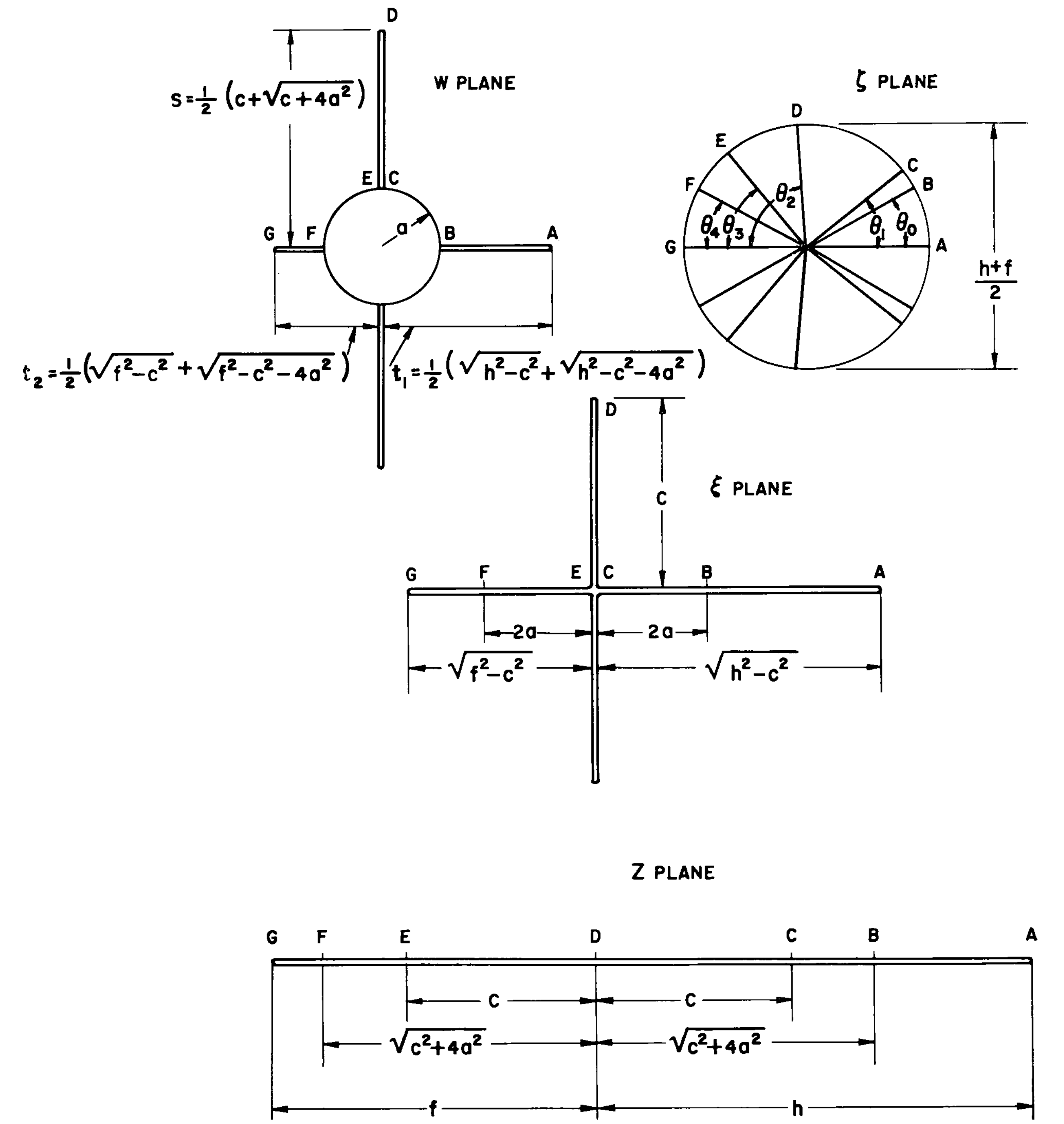

FIG. 3. Conformal mapping of cross section onto a circle.

and

$k \quad = \tan\frac{\lambda}{2}\tan\frac{\mu}{2}$ (note $0 \le k \le 1$, since $0 \le \lambda \le \pi$, $0 \le \mu \le \pi - \lambda$)

$k' \quad = \sqrt{1 - k^2}$

$E(\varphi, k) = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2 \varphi}\,(d\varphi$ = incomplete elliptic integral of second kind)

$E_0(k) = E\left(\frac{\pi}{2}, k\right)$ = complete elliptic integral of second kind

$F(\varphi, k) = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}$ = incomplete elliptic integral of first kind

$K(k) = F\left(\frac{\pi}{2}, k\right)$ = complete elliptic integral of first kind

A graph of the function $G(\lambda,\mu)$ is shown in Fig. 5, and a table of values of the function is given in Table 1. A discussion of the function is given in Appendix (1).

The expressions for A_{13} and A_{33} cannot be given in terms of tabulated functions. The expression for A_{33} will not be given here, but A_{13} is worked out in Appendix (2) The result is

$$A_{13} = \frac{1}{\pi}\left(\frac{2a}{D}\right)^2 \frac{h+f}{2D}(\sin\theta_0 + \sin\theta_1 - \sin\theta_3 - \sin\theta_4) - \frac{1}{\pi}\left(\frac{h+f}{2D}\right)^3 \Bigg\{ 2(\sin\theta_0 - \sin\theta_1 + \sin\theta_3 - \sin\theta_4) -$$
$$\frac{2}{3}(\sin^3\theta_0 - \sin^3\theta_1 + \sin^3\theta_3 - \sin^3\theta_4) - (\cos\theta_0 + \cos\theta_1 - \cos\theta_3 - \cos\theta_4) \times$$
$$\left[\frac{1}{2}(\theta_0 - \theta_1 - \theta_3 + \theta_4) + \frac{1}{4}(\sin 2\theta_0 - \sin 2\theta_1 - \sin 2\theta_3 + \sin 2\theta_4)\right] +$$
$$\frac{\pi}{8}(\cos\theta_0 + \cos\theta_1 + \cos\theta_3 + \cos\theta_4)(\cos\theta_0 - \cos\theta_1 + \cos\theta_3 - \cos\theta_4) \times$$
$$(\cos\theta_0 - \cos\theta_1 - \cos\theta_3 + \cos\theta_4) + 2H(\theta_0, \theta_1, \theta_3, \theta_4) + 2H(\theta_1, \theta_0, \theta_4, \theta_3)\Bigg\} \quad (36)$$

where

$$\left.\begin{aligned} H(\alpha, \beta, \gamma, \delta) &= H^*(\alpha, \beta, \gamma, \delta) + H^*(\delta, \gamma, \beta, \alpha) \\ H^*(\alpha, \beta, \gamma, \delta) &= \int_0^{\alpha} \sqrt{(\cos\theta - \cos\alpha)(\cos\theta - \cos\beta)(\cos\theta + \cos\gamma)(\cos\theta + \cos\delta)}\,\cos\theta\, d\theta \end{aligned}\right\} \quad (37)$$

The value of H^* can easily be determined numerically for any particular cross section, since the integrand has no singularities.

The Stability Derivatives for a Wing-Body-Vertical-Tail Configuration

The stability derivatives for the equilibrium condition $\alpha = \alpha, \beta = p = q = r = \dot{\alpha} = \dot{\beta} = \dot{p} = \dot{q} = \dot{r} = \dot{U} = 0$ can be obtained by substituting the expressions for the inertia coefficients into Eqs. (13) through (27).

In general, the integrals involving A_{11} and A_{13} will have to be integrated numerically. The integrals involving A_{22} can often be integrated in simple forms because of the simplicity of the analytic expression for A_{22}.

It is interesting that for $\alpha = 0$ the only stability derivatives that depend upon the distribution of the inertia coefficients with x are C_{m_α}, C_{n_β}, C_{n_p}, C_{n_r}, C_{m_q}, and all derivatives with respect to $\dot{\alpha}$, $\dot{\beta}$, $\dot{p}$, $\dot{q}$, $\dot{r}$, and $\dot{U}$. All the other stability derivatives depend only on the values of the inertia coefficients in the base plane. Furthermore, the important quantities $C_{n_r} - C_{n_{\dot{\beta}}}$, and $C_{m_q} + C_{m_{\dot{\alpha}}}$ depend only on the values of A_{11} and A_{22} in the base plane; in fact,

$$C_{n_r} - C_{n_{\dot{\beta}}} = -2\left(\frac{x_b}{D}\right)^2 \bar{A}_{11} = \left(\frac{x_b}{D}\right)^2 C_{Y_\beta}$$

$$C_{m_q} + C_{m_{\dot{\alpha}}} = -2\left(\frac{x_b}{D}\right)^2 \bar{A}_{22} = \left(\frac{x_b}{D}\right)^2 C_{Z_\alpha}$$

This means, for example, that the damping of the short-period longitudinal oscillation depends only on the geometry of the base plane. If τ is the time to damp to $1/e$ times the undamped amplitude, then

$$\tau = \frac{4W}{g\rho U \dfrac{\pi D^2}{4}\left[-C_Z^{\alpha} - \left(\dfrac{D}{r}\right)^2 (C_{m_q} + C_{m_{\dot{\alpha}}})\right]}$$

$$\tau = \frac{2W}{g\rho U \pi \bar{s}^2 \left(1 - \dfrac{\bar{a}^2}{\bar{s}^2} + \dfrac{\bar{a}^4}{\bar{s}^4}\right)\left(1 + \dfrac{x_b^2}{r^2}\right)}$$

where W = weight of the missile, r = radius of gyration of the missile about the y-axis, and g = acceleration due to gravity.

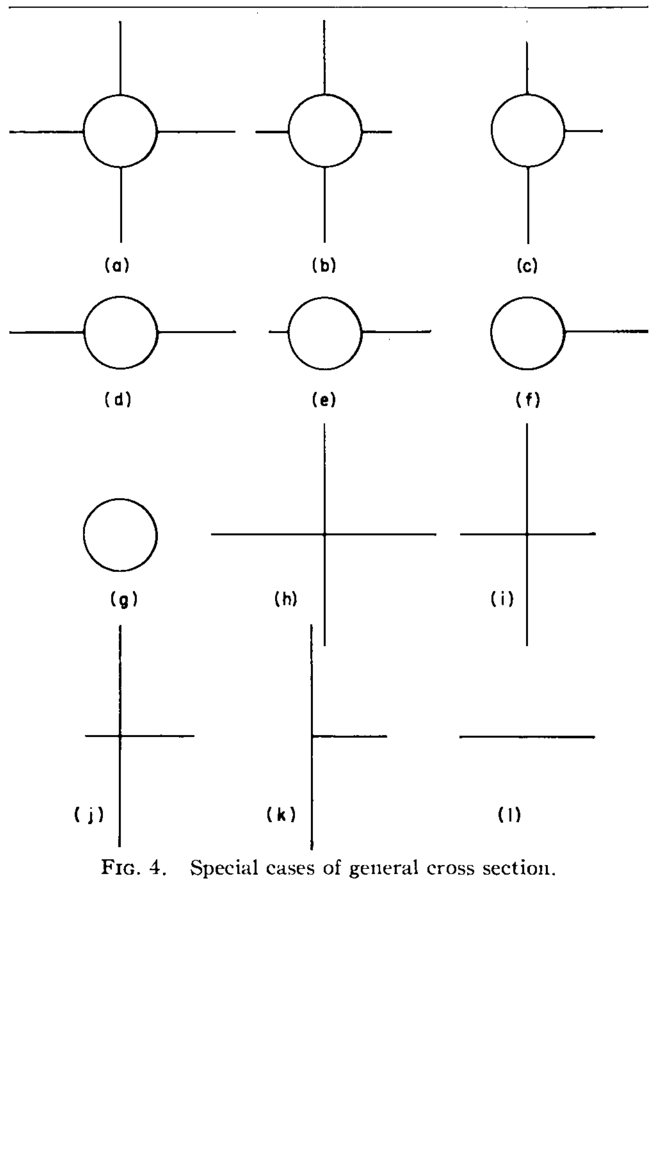

Fig. 4. Special cases of general cross section.

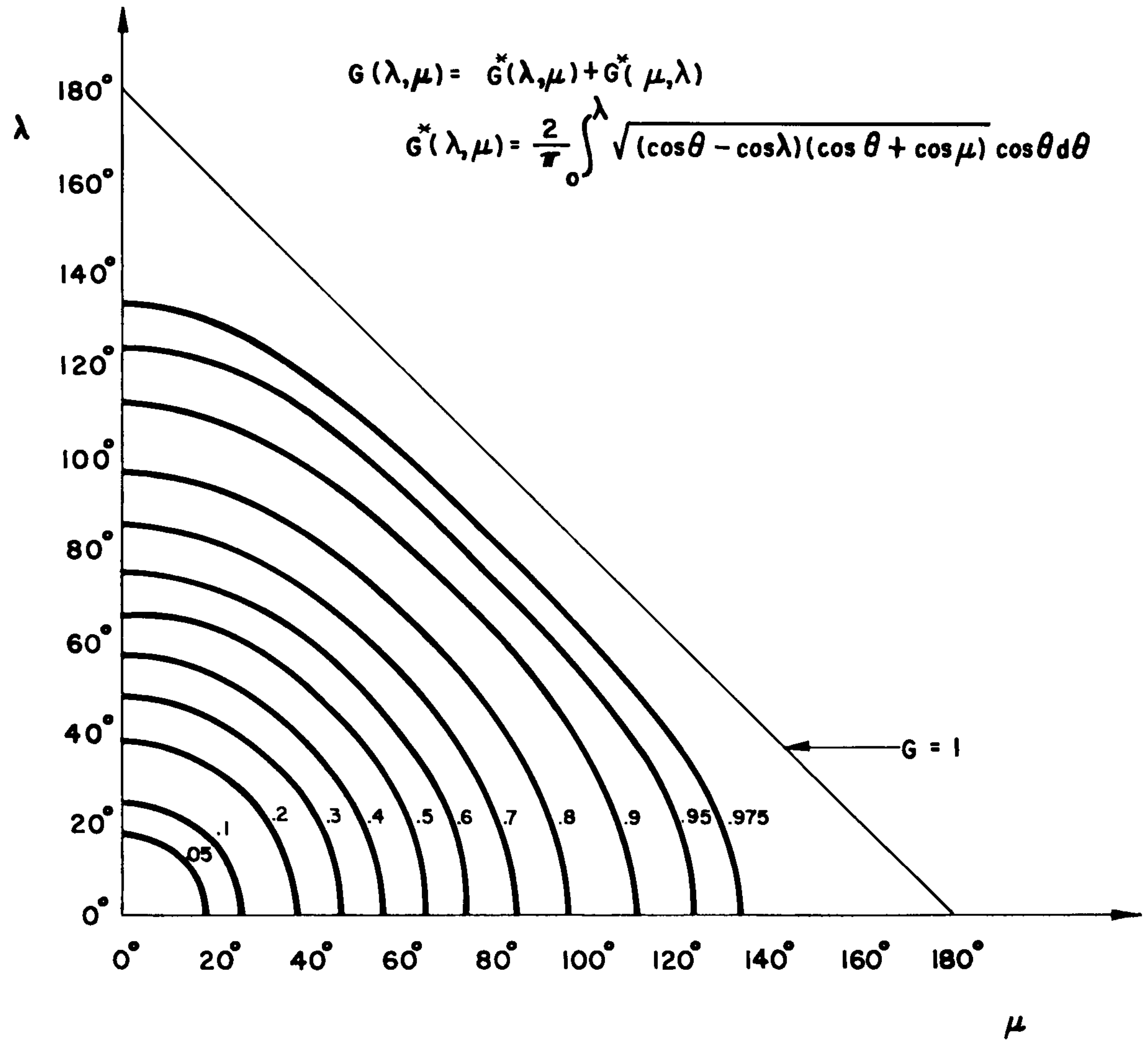

FIG. 5. Plot of $G(\lambda,\mu)$.

Appendix (1)

INERTIA COEFFICIENT A_{11}

Referring to Fig. 3, we wish to find the inertia coefficient (apparent additional mass coefficient) for motion of the cross section in the w_2 direction where $w = w_1 + iw_2$. The velocity potential for unit velocity in this direction is

$$\varphi_1 = Rl\{-i(\zeta - w) + i[(h + f)^2/16\zeta]\} \tag{1}$$

where

$$\left[\frac{h - f}{2} + \zeta + \frac{(h + f)^2}{16\zeta}\right]^2 = c^2 + \left(w + \frac{a^2}{w}\right)^2$$

On the boundary of the cross section $\zeta = [(h + f)/4]e^{i\theta}$, so the potential there is

$$\varphi_1 = [(h + f)/2]\sin\theta - w_2$$

$$w_1 + iw_2 = \pm\frac{h + f}{4}[\sqrt{(\cos\theta - \cos\theta_0)(\cos\theta + \cos\theta_4)} + \sqrt{(\cos\theta - \cos\theta_1)(\cos\theta + \cos\theta_3)}] \tag{2}$$

The inertia coefficient is given by

$$A_{11} = -\frac{1}{S}\oint \varphi_1\frac{\partial\varphi_1}{\partial n}ds = -\frac{4}{\pi D^2}\oint \varphi_1 dw_1 \tag{3}$$

TABLE 1

The Function $G(\lambda,\mu) = G(\mu,\lambda)$

λ/μ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
0°	0									
10	0.015	0.030								
20	0.059	0.074	0.117							
30	0.129	0.143	0.185	0.250						
40	0.220	0.234	0.273	0.334	0.413					
50	0.325	0.338	0.374	0.431	0.503	0.586				
60	0.437	0.449	0.482	0.534	0.599	0.673	0.750			
70	0.549	0.560	0.590	0.635	0.693	0.757	0.823	0.883		
80	0.655	0.665	0.690	0.730	0.779	0.833	0.886	0.933	0.969	
90	0.750	0.757	0.779	0.812	0.853	0.897	0.937	0.970	0.992	1.0
100	0.829	0.836	0.853	0.880	0.912	0.945	0.973	0.992	1.0	
110	0.891	0.897	0.910	0.931	0.955	0.977	0.993	1.0		
120	0.937	0.942	0.951	0.966	0.982	0.994	1.0			
130	0.968	0.971	0.978	0.987	0.995	1.0				
140	0.986	0.988	0.992	0.996	1.0					
150	0.995	0.996	0.998	1.0						
160	0.999	0.999	1.0							
170	0.999	1.0								
180	1.0									

From Eq. (2) we see that

$$w_1 = \pm\frac{h+f}{4}\begin{cases}\sqrt{(\cos\theta-\cos\theta_0)(\cos\theta+\cos\theta_4)}+\sqrt{(\cos\theta-\cos\theta_1)(\cos\theta+\cos\theta_3)}; & 0\le\theta\le\theta_0,\ \pi-\theta_4\le\theta\le\pi\\ \sqrt{(\cos\theta-\cos\theta_1)(\cos\theta+\cos\theta_3)}; & \theta_0<\theta<\theta_1;\ \pi-\theta_3\le\theta\le\pi-\theta_4\\ 0; & \theta_1\le\theta\le\theta_3\end{cases} \tag{4}$$

Also, it is easily seen that

$$\oint w_2\, dw_1 = \pi a^2 \tag{5}$$

Substituting Eqs. (2) and (4) into Eq. (3), integrating by parts, and changing integration variables in the last two integrals, we have

$$A_{11} = \frac{1}{\pi}\left(\frac{h+f}{D}\right)^2\left[\int_0^{\theta_0}\sqrt{(\cos\theta-\cos\theta_0)(\cos\theta+\cos\theta_4)}\cos\theta\,d\theta + \int_0^{\theta_1}\sqrt{(\cos\theta-\cos\theta_1)(\cos\theta+\cos\theta_3)}\cos\theta\,d\theta + \int_0^{\theta_3}\sqrt{(\cos\theta-\cos\theta_3)(\cos\theta+\cos\theta_1)}\cos\theta\,d\theta + \int_0^{\theta_4}\sqrt{(\cos\theta-\cos\theta_4)(\cos\theta+\cos\theta_0)}\cos\theta\,d\theta\right] - 2\left(\frac{a}{D}\right)^2 \tag{6}$$

Hence,

$$A_{11} = \frac{1}{2}\left(\frac{h+f}{D}\right)^2[G(\theta_0,\theta_4)+G(\theta_1,\theta_3)] - 2\left(\frac{a}{D}\right)^2 \tag{7}$$

where

$$\left.\begin{aligned} G(\lambda,\mu) &= G^*(\lambda,\mu) + G^*(\mu,\lambda)\\ G^*(\lambda,\mu) &= \frac{2}{\pi}\int_0^{\lambda}\sqrt{(\cos\theta-\cos\lambda)(\cos\theta+\cos\mu)}\cos\theta\,d\theta\end{aligned}\right\} \tag{8}$$

where $0\le\lambda\le\pi$ and $0\le\mu\le\pi-\lambda$.

The function $G(\lambda,\mu)$ can be expressed in terms of complete and incomplete elliptic integrals by a lengthy manipulation, leading to Eq. (9) in the text.†

Note that $G(\lambda,\mu) = G(\mu,\lambda)$. If $\mu = 0$, then $k = 1$, $k' = 0$, and

$$G(\lambda,0) = 1 - \cos^4(\lambda/2) \tag{9}$$

If $\mu = \pi - \lambda$, then $k = 1$, $k' = 0$, and

$$G(\lambda, \pi - \lambda) = 1 \tag{10}$$

If $\lambda = \mu$, then

$$G(\lambda,\lambda) = \sin^2\lambda \tag{11}$$

It is easy to show that the contour lines of constant G have the following slopes (see Fig. 5):

$$d\lambda/d\mu = \begin{Bmatrix} 0; & \mu = 0 \\ -1; & \lambda = \mu \\ \infty; & \lambda = 0 \end{Bmatrix} \tag{12}$$

† The author is indebted to H. A. Linstone for working out this expression.

Two particular cases are easily worked out in elementary functions. (a) $t_1 = t_2 = t$, corresponding to Fig. 4b:

$$A_{11} = \left(\frac{2t}{D}\right)^2\left(1 - \frac{a^2}{t^2} + \frac{a^4}{t^4}\right) \tag{13}$$

which checks Spreiter's result of Eq. (7).

(b) $a = 0$, $t_2 = 0$, corresponding to Fig. 4k:

$$A_{11} = \left(\frac{t_1}{D}\right)^2\left(1 - 2\frac{s^2}{t_1^2} + 2\frac{s}{t_1}\sqrt{1 + \frac{s^2}{t_1^2}}\right) \tag{14}$$

$A_{11} = \left(\frac{t_1}{D}\right)^2$ as $\frac{t_1}{s} \to \infty$, corresponding to a flat plate

$A_{11} = 2\left(\frac{t_1}{D}\right)^2$ as $\frac{s}{t_1} \to \infty$, corresponding to a flat plate in presence of an infinite end plate on one end

Appendix (2)

Inertia Coefficient A_{13}

The expression for A_{13} is

$$\left.\begin{aligned} A_{13} &= -(1/SD) \oint \varphi_1(\partial\varphi_3/\partial n)\, ds \\ A_{13} &= -\frac{8}{\pi D^3}\left(-\int_{\theta=0}^{\theta_0} \varphi_1 w_1\, dw_1 + \int_{\theta=\theta_1}^{\pi-\theta_3} \varphi_1 w_2\, dw_2 - \int_{\theta=\pi-\theta_4}^{\pi} \varphi_1 w_1\, dw_1\right) \end{aligned}\right\} \tag{1}$$

where

$$\left.\begin{aligned} \varphi_1 &= [(h+f)/2]\sin\theta - w_2 \\ w_1^2 &= \frac{1}{4}\left(\frac{h+f}{2}\right)^2\left[\sqrt{(\cos\theta - \cos\theta_0)(\cos\theta + \cos\theta_4)} + \sqrt{(\cos\theta - \cos\theta_1)(\cos\theta + \cos\theta_3)}\right]^2 \\ &\text{for } 0 \le \theta \le \theta_0 \text{ and } \pi - \theta_4 \le \theta \le \pi; \\ w_2^2 &= \frac{1}{4}\left(\frac{h+f}{2}\right)^2\left[\sqrt{(\cos\theta - \cos\theta_0)(\cos\theta + \cos\theta_4)} + \sqrt{(\cos\theta_1 - \cos\theta)(\cos\theta + \cos\theta_3)}\right]^2 \\ &\text{for } \theta_1 \le \theta \le \pi - \theta_3. \end{aligned}\right\} \tag{2}$$

Substituting Eq. (2) into Eq. (1), integrating by parts, and manipulating, we finally arrive at Eq. (10) of the text.

For the special case of no body—i.e., $a = 0$—Eq. (10) simplifies greatly, since $\theta_0 = \theta_1$ and $\theta_3 = \theta_4$ for this case:

$$A_{13} = -\frac{1}{2\pi}\left(\frac{h+f}{D}\right)^3\Bigg\{\sin\theta_0 + \sin\theta_4 - \frac{1}{3}\sin^3\theta_0 - \frac{1}{3}\sin^3\theta_4 + (\cos\theta_4 - \cos\theta_0)\left[\frac{1}{2}(\theta_0 - \theta_4) + \frac{1}{4}(\sin 2\theta_0 - \sin 2\theta_4)\right] - \cos\theta_0\cos\theta_4(\sin\theta_0 + \sin\theta_4)\Bigg\} \tag{3}$$

References

[1] Munk, Max M., *The Aerodynamic Forces on Airship Hulls*, N.A.C.A. Report No. 184, 1924.

[2] Jones, Robert T., *Properties of Low Aspect Ratio Pointed Wings at Speeds Below and Above the Speed of Sound*, N.A.C.A. Report No. 835, 1946.

[3] Spreiter, John R., *The Aerodynamic Forces on Slender Plane- and Cruciform-Wing and Body Combinations*, N.A.C.A. Report No. 962, 1950.

[4] Lamb, Horace, *Hydrodynamics*, 6th Ed. pp. 160–168; Dover Publications, New York.

[5] Ribner, Herbert S., *The Stability Derivatives of Low-Aspect Ratio Triangular Wings at Subsonic and Supersonic Speeds*, N.A.C.A. T.N. No. 1423, September, 1947.